University of California, Berkeley
Physics 110A Spring 2003 Section 2 (*Strovink*)

Solution Set 2

1. We use integration by parts,

$$\int_{-\infty}^{\infty} dx f(x)\frac{d}{dx}\theta(x-x') = f(x)\theta(x-x')|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\frac{df(x)}{dx}\theta(x-x') = f(\infty) - \int_{x'}^{\infty} dx\frac{df(x)}{dx} = f(x') \quad (1)$$

2. (a) Using the $BAC - CAB$ identity,

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}, \quad (2)$$

and the Coulomb Gauge condition, $\nabla\cdot\vec{A} = 0$, Ampere's Law becomes,

$$\nabla\times\vec{B} = -\nabla^2\vec{A} = \mu_0\vec{J}. \quad (3)$$

(b) The Green function for the operator ∇^2 obeys the relation,

$$\nabla^2 G(\vec{r}-\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}'). \quad (4)$$

Thus, we must have that,

$$\frac{\mu_0}{4\pi}\nabla^2\int d^3\vec{r}' G(\vec{r}-\vec{r}')\vec{J}(\vec{r}') = -\mu_0\int d^3\vec{r}'\delta(\vec{r}-\vec{r}')\vec{J}(\vec{r}') = -\mu_0\vec{J}(\vec{r}). \quad (5)$$

Since, as shown in Griffiths equation 1.102, $G(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$, we see that Ampere's Law is solved by,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}'\,\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad (6)$$

3. +

4. We consider, in this problem, the effect of a hypothetical photon mass on the question of whether the electric field inside a conducting shell vanishes.

(a) Suppose that the photon mass is negligible. Then, as the inner shell is an isolated, uncharged conductor, the region within the outer conductor has no net enclosed charge, and from section 2.5 of Griffiths, we know that the electric field there must vanish. Since $\vec{E} = 0 = -\nabla\phi(r,t)$, we see that we have that $\phi_u(r,t)$ is independent of r inside the outer shell, and as we know that $\phi_u(R_2,t) = V_0\cos\omega t$, we have that,

$$\phi_u(r,t) = V_0\cos\omega t \quad (7)$$

(b) Now, suppose that $\overline{\lambda}^{-2}$ is small. Then, we can approximate $\phi(r,t)\approx\phi_u(r,t)$. By spherical symmetry, it is clear that $\vec{E}$ must only be a function of r and the potential on each of the spherical conducting shells must be constant, so $\vec{E}$ must be radially directed. Now, let's integrate the modified Gauss' Law,

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} - \frac{\phi}{\overline{\lambda}^{-2}}. \quad (8)$$

over a ball B of radius r (where $R_2 > r > R_1$) bounded by the spherical shell S and use the Divergence Theorem to get,

$$\begin{aligned}\int_B \nabla\cdot\vec{E}d\tau = \int_S \vec{E}\cdot da = 4\pi r^2 E_r(r,t) &= \int_B \frac{\rho}{\epsilon_0}d\tau + \overline{\lambda}^{-2}\int_B \phi_u(r,t)d\tau = 0 - \overline{\lambda}^{-2}\int_B V_0\cos\omega t d\tau \\ &= -\frac{4}{3\overline{\lambda}^2}\pi r^3 V_0\cos\omega t.\end{aligned} \quad (9)$$

Thus, we can solve for $E_r(r,t)$ to find that,

$$E_r(r,t) = -\frac{V_0 r\cos\omega t}{3\overline{\lambda}^2}. \quad (10)$$

(c) Now, the potential difference between the outer and inner spheres can be calculated by integrating,

$$v(t) = \phi(R_2, t) - \phi(R_1, t) = -\int_{R_1}^{R_2} E_r(r, t)dr = \int_{R_1}^{R_2} \frac{V_0 r \cos\omega t}{3\bar{\lambda}^2} dr = \frac{V_0 \cos\omega t}{3\bar{\lambda}^2}(R_2^2 - R_1^2). \tag{11}$$

(d) Suppose $R_2 = 1.5m$ and $R_1 = 0.5m$ and the amplitude of $v(t)$, (which is just the coefficient of $\cos\omega t$ in equation (11)) is found to be $10^{-15}V_0$, then we see that,

$$10^{-15}V_0 = V_0 \times \frac{(1.5m)^2 - (0.5m)^2}{3\bar{\lambda}^2} \tag{12}$$

$$\Rightarrow \lambda^{-2} = \frac{3}{2} \times 10^{-15} m^{-2} \tag{13}$$

Now, using the fact that $\bar{\lambda} = \frac{\hbar}{m_0 c}$,

$$m_0 = \frac{\hbar}{c}\sqrt{\frac{3}{2} \times 10^{-15} m^{-1}} = \frac{6.58 \times 10^{-16} eVs \times 3 \times 10^8 m/s}{c^2}\sqrt{\frac{3}{2} \times 10^{-15} m} = 7.64 \times 10^{-17} eV/c^2. \tag{14}$$

As the mass of the Z^0 boson is $9.1 \times 10^{10} eV/c^2$, we see that the ratio of the photon mass to the Z^0 mass is given by,

$$7.64 \times 10^{-17} / 9.1 \times 10^{10} = 8.4 \times 10^{-28}. \tag{15}$$

5. (a) First, we integrate Gauss's Law over a ball B_r of radius $r > b$ bounded by a sphere S_r,

$$\int_{S_r} E \cdot da = 4\pi r^2 E_r = \int_{B_r} \frac{\rho(r)}{\epsilon_0} d\tau = 4\pi \int_a^b dr r^2 \frac{k}{\epsilon_0 r^2} = \frac{4\pi k(b-a)}{\epsilon_0} \Rightarrow E_r(r > b) = \frac{k(b-a)}{\epsilon_0 r^2}. \tag{16}$$

Doing the same thing for $b > r > a$, we get

$$4\pi r^2 E_r = \int_{B_r} \frac{\rho(r)}{\epsilon_0} d\tau = 4\pi \int_a^r dr r^2 \frac{k}{\epsilon_0 r^2} = \frac{4\pi k(r-a)}{\epsilon_0} \Rightarrow E_r(b > r > a) = \frac{k(r-a)}{\epsilon_0 r^2}, \tag{17}$$

and for $r < a$ we find that $E_r(r < a) = 0$ as the charge density vanishes there. To find V, we simply integrate this result from ∞ to the origin,

$$\begin{aligned} V(0) - V(\infty) = V(0) &= \int_0^\infty E_r dr = \int_0^a E_r(r < a)dr + \int_a^b E_r(b > r > a)dr + \int_b^\infty E_r(r > b)dr \\ &= \frac{k}{\epsilon_0}\left(\int_a^b \frac{r-a}{r^2} dr + \int_b^\infty \frac{b-a}{r^2}\right) = \frac{k}{\epsilon_0} \ln\frac{b}{a}. \end{aligned} \tag{18}$$

(b) Now, we do the same problem using Griffiths Equation 2.29

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau, \tag{19}$$

which becomes, in our case,

$$V(0) = \frac{1}{\epsilon_0} \int \frac{\rho(r')}{r'} r'^2 dr' = \frac{k}{\epsilon_0} \int_a^b \frac{dr}{r} = \frac{k}{\epsilon_0} \ln\frac{b}{a}. \tag{20}$$

This is clearly the easier way to do this problem!

6. We use Griffiths Equation 2.29 to compute the potential at a point on the z-axis above the disk. As we computed in problem 7(a) of Problem Set 1, the charge density for this disk is given in cylindrical coordinates by,

$$\rho(\vec{r}') = \sigma\theta(b - r')\delta(z'), \tag{21}$$

where σ is the surface charge density on the disk and b is its radius. Now, we compute the potential at a point on the z axis above the disk,

$$V(z) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\phi' \int_0^{\infty}\int_{-\infty}^{\infty} \frac{\sigma\theta(b-r')\delta(z')}{\sqrt{r'^2+z^2}} r'dzdr' = \frac{\sigma}{4\pi\epsilon_0}2\pi\int_0^b \frac{r'}{\sqrt{r'^2+z^2}}dr' = \frac{\sigma}{4\pi\epsilon_0}2\pi\left(\sqrt{b^2+z^2}-z\right). \tag{22}$$

By cylindrical symmetry, we expect that only E_z is non-zero on the z axis, and we find that

$$E_z(z) = -\frac{\partial V}{\partial z} = \frac{\sigma}{4\pi\epsilon_0}2\pi\left(1-\frac{z}{\sqrt{b^2+z^2}}\right) \tag{23}$$

Now, let us compute the solid angle subtended by the disk, as seen from the point on the z-axis we are considering. Recall that the solid angle subtended by some object is just the area that it appears to cover when projected on a sphere of unit radius away from the viewer. For example, the sky above us covers a hemisphere and would subtend a solid angle of 2π, while the moon covers a much smaller solid angle. In fact, we can use the computation we will do below, given the distance to the moon (z), and its radius (b) to compute it. It is easiest to do this computation by considering spherical coordinates centered on that point. In these coordinates, the disk looks like it is a distance z below the xy-plane, and in terms of θ and ϕ, it occupies a region specified by $\theta \in [\arctan\frac{-b}{z}, \pi]$ and $\phi \in [0, 2\pi]$. To calculate the solid angle corresponding to this,

$$\Omega(z) = \int_0^{2\pi} d\phi \int_{\arctan\frac{-b}{z}}^{\pi} \sin\theta d\theta = 2\pi\int_{-1}^{\frac{z}{\sqrt{b^2+z^2}}} d(\cos\theta) = 2\pi\left(1-\frac{z}{\sqrt{b^2+z^2}}\right) \tag{24}$$

Comparing this expression with our answer for $E_z(z)$, we see that $E_z \propto \Omega$.

7. First, we consider the question for an infinite cylindrical cylinder (so $l \to \infty$). Then, by symmetry, it is clear that the electric field gets opposite contributions from the points above and below it in the cylinder, and therefore vanishes. If the cylinder is finite, and we consider the origin, this argument still works, and the field must vanish by symmetry. However, for all other points on the z-axis (we will, without loss of generality, consider those points above 0, $z > 0$), there is some portion of the cylinder below it which no longer gets a cancelling contribution from above it. In particular, for a point a distance z above the origin, the segment of length $2z$ beginning a distance of $l - z$ below the point gives a non-zero contribution to the electric field. Now, let us attempt to calculate the electric field due to this segment of the cylinder using Griffiths (2.8) in cylindrical coordinates, where this segment is just the region $\rho \in [0, b]$, $z \in [-l/2, -l/2+2z]$. First, note that by cylindrical symmetry, the components of the electric field in the r and ϕ directions must vanish (the contributions from opposite sides of the segment of cylinder cancel eachother). So, we only consider the z component of Griffiths (2.8), which is,

$$\begin{aligned} E_z(z) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(z-z')d\tau' = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\phi' \int_{-\frac{l}{2}}^{-\frac{l}{2}+2z}\int_0^b \frac{\rho_0}{(r'^2+(z-z')^2)^{3/2}} r'(z-z')dr'dz' \\ &= \frac{\rho_0}{2\epsilon_0}\int_{-\frac{l}{2}}^{-\frac{l}{2}+2z}\left(1-\frac{(z-z')}{\sqrt{b^2+(z-z')^2}}\right)dz' = \frac{\rho_0}{2\epsilon_0}\left(2z+\sqrt{b^2+(l/2-z)^2}-\sqrt{b^2+(l/2+z)^2}\right) \end{aligned} \tag{25}$$

As we assume that $b \ll l \Rightarrow \frac{b}{l} \ll 1$ and $|z| < l/4$, $(l/2 \pm z)$ is always on the order of l in the above integral and so the b^2 term in the square root will always be very small comparted to the $(l/2 \pm z)^2$ term. Thus, using the Taylor approximation,

$$(1+x)^{1/2} = \sum_{n=0}^{\infty}\frac{x^n}{n!}\frac{d^n}{dx^n}(1+x)^{1/2}\bigg|_{x=0} = 1+\frac{x}{2}+\cdots \tag{26}$$

we make the approximation that,

$$\sqrt{b^2+(l/2\pm z)^2} = (l/2\pm z)\left(1+\frac{b^2}{(l/2\pm z)^2}\right)^{1/2} \approx (l/2\pm z)+\frac{b^2}{2(l/2\pm z)}+\ldots \tag{27}$$

where the ... stands for terms which are proportional to higher powers of $\frac{b^2}{(l/2\pm z)^2}$ which we can neglect as they are always small. Now, plugging this into the above expression, we have,

$$\begin{aligned} E_z(z) &\approx \frac{\rho_0}{2\epsilon_0}\left(2z + (l/2 - z) + \frac{b^2}{2(l/2-z)} - (l/2+z) - \frac{b^2}{2(l/2+z)}\right) \\ &= \frac{\rho_0 b^2}{4\epsilon_0}\left(\frac{1}{l/2-z} - \frac{1}{l/2+z}\right) = \frac{\rho_0 b^2 z}{4\epsilon_0(l^2/4 - z^2)}. \end{aligned} \tag{28}$$

8. First, we note that (being careful with taking derivatives of $\frac{1}{r}$),

$$\begin{aligned} -\epsilon_0\nabla^2 V(r) = \rho(r) &= -\epsilon_0\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) \propto -\frac{1}{r^2}\frac{\partial}{\partial r}\left(-kre^{-kr} + e^{-kr}r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) \\ &= \frac{ke^{-kr}}{r^2} - \frac{k^2e^{-kr}}{r} - \frac{ke^{-kr}}{r^2} - e^{-kr}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) \\ &= -\frac{k^2e^{-kr}}{r} - e^{-kr}\nabla^2\left(\frac{1}{r}\right) = 4\pi e^{-kr}\delta^3(\vec{r}) - \frac{k^2e^{-kr}}{r}. \end{aligned} \tag{29}$$

Now, to find the total charge, we need to evaluate the integral,

$$Q = \int \rho(r)d\tau \propto \left(\int 4\pi e^{-kr}\delta^3(\vec{r})d\tau - 4\pi k^2\int_0^\infty re^{-kr}dr\right) = \left(4\pi - 4\pi\int_0^\infty xe^{-x}dx\right) = 0, \tag{30}$$

where we've used the fact that $\int_0^\infty xe^{-x}dx = 1$. Essentially, the charge density at the origin vanishes due to a cancellation between the delta function (which represents the unscreened bare charge at the origin) and the negative charges which conglomerate near the origin to screen it.